Defining zero-crossings to verify hybrid synchronous programs

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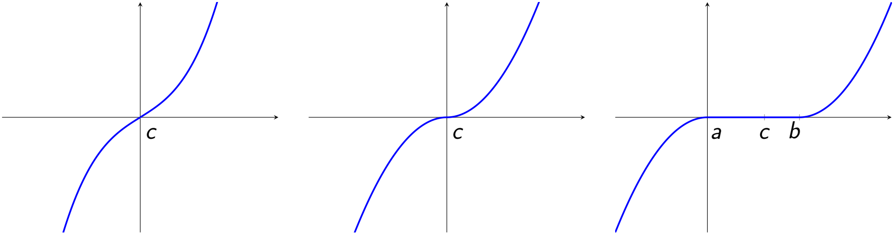
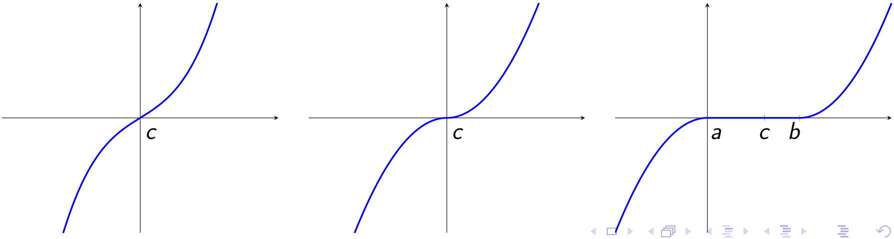
What is a zero-crossing (informally)?

- A zero-crossing is when a function f from $\mathbb{R}$ to $\mathbb{R}$ changes sign
- We limit ourselves to negative-to-positive, the converse is symmetric
- If f is identically zero for some time > 0, where is the zero-crossing?



Why do we care?

- Zero-crossings are used by hybrid systems languages such as **Zélus** and **Simulink** to interface from continuous to discrete
- It seems simple enough, but there are some tricky corner cases
- A formal definition is needed to understand how to verify hybrid and continuous Zélus programs
- Specifically, needed for a precise semantics of the discrete/continuous interface, and for soundness of verification rules
- In this work we try to **detach ourselves from the algorithm** that finds the zero-crossings, and instead build an analysis-based solution



Notation

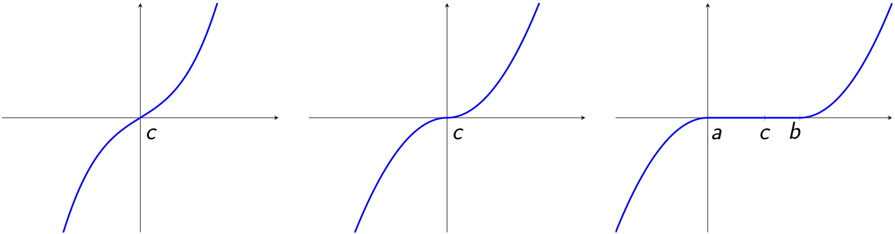
We are mostly interested about **closed intervals of** $\mathbb{R}$. To unify notations, let us define (does anyone know of a standard notation for this?). For any $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, we write $(\!|a; b|\!)$ for:

- $(\!|a; b|\!) = [a; b]$ if $a \in \mathbb{R}$ and $b \in \mathbb{R}$
- $(\!|a; b|\!) = (-\infty; b]$ if $a = -\infty$ and $b \in \mathbb{R}$
- $(\!|a; b|\!) = [a; +\infty)$ if $a \in \mathbb{R}$ and $b = +\infty$
- $(\!|a; b|\!) = (-\infty; +\infty)$ if $a = -\infty$ and $b = +\infty$

Note that $(\!|a; b|\!) \subseteq \mathbb{R}$ and cannot include $-\infty$ or $+\infty$.

Basic hypotheses and setup

- We limit ourselves to **continuous** functions (for now)
- Let f be a function defined and continuous on $(\!|\ell; u|\!)$ with $\ell \in \{-\infty\} \cup \mathbb{R}$, $u \in \mathbb{R} \cup \{+\infty\}$ and $\ell \leq u$
- Let $c \in (\!|\ell; u|\!)$ **such that** $f(c) = 0$, note that $c \in \mathbb{R}$
- Question: is c a zero-crossing?
- If not, is there a zero-crossing "close to" c and how to define it?



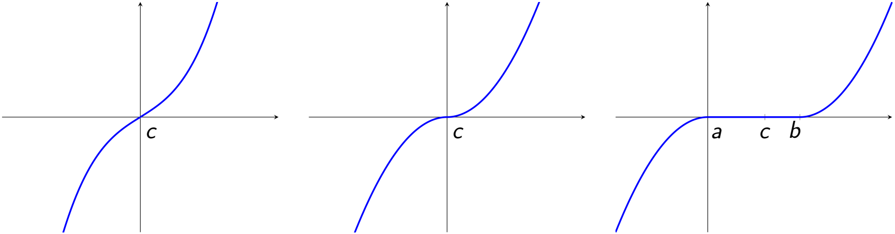
Defining a and b

From c define a and b as:

- $a = \inf\{a \in \llparenthesis \ell; u \rrparenthesis \mid \forall x \in \llparenthesis a; c], f(x) = 0\}$
- $b = \sup\{b \in \llparenthesis \ell; u \rrparenthesis \mid \forall x \in [c; b \rrparenthesis, f(x) = 0\}$
- note that $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$

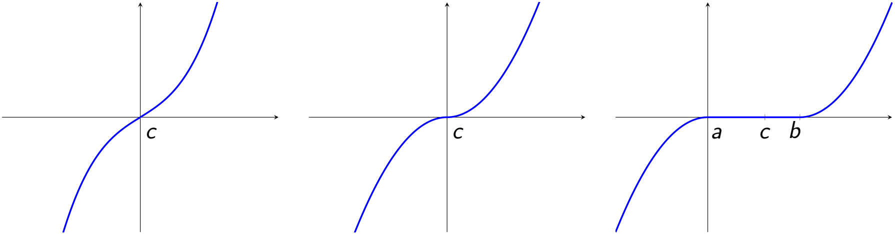
Research question:

Given $z \in \llparenthesis a; b \rrparenthesis$, what does it mean for z to be a zero-crossing?



Some properties of a and b

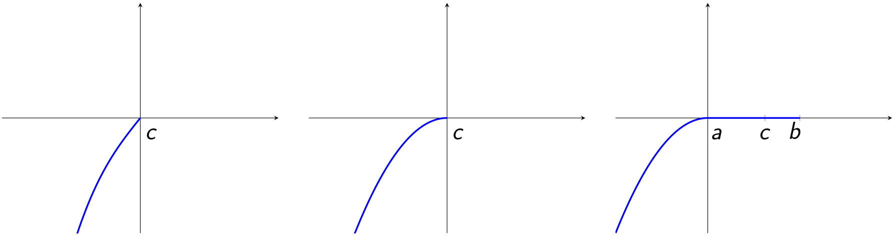
- $a = \inf\{a \in \llparenthesis \ell; u \rrparenthesis \mid \forall x \in \llparenthesis a; c], f(x) = 0\}$
- $b = \sup\{b \in \llparenthesis \ell; u \rrparenthesis \mid \forall x \in [c; b \rrparenthesis, f(x) = 0\}$
- $a = b \in \mathbb{R}$ always possible (nominal case $a = b = c$)
- $\forall x \in \llparenthesis a; b \rrparenthesis, f(x) = 0$
- $\llparenthesis a; b \rrparenthesis \subseteq \llparenthesis l; u \rrparenthesis$, i.e. $\ell \leq a \leq b \leq u$
- if $a > l \geq -\infty$, then $\forall \varepsilon > 0, \exists x \in [a - \varepsilon; a), f(x) \neq 0$
- if $b < u \leq +\infty$, then $\forall \varepsilon > 0, \exists x \in (b; b + \varepsilon], f(x) \neq 0$



Case distinction: 7 possibilities for $x < a$

Case 1: f is strictly negative in a ball left of a. Formally:
$\ell < a$ and $\exists \eta > 0, \forall x \in [a - \eta, a), f(x) < 0$

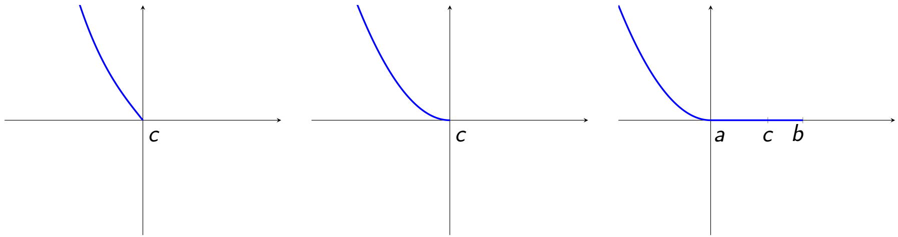
- examples: $(x-a)^{2k+1}$, $-|x-a|^k$, $-\exp\left(-\frac{1}{(x-a)^2}\right)$
- essentially $\lim_{x \to a^-} f(x) = 0^-$



Case distinction: 7 possibilities for $x < a$

Case 2: f is strictly positive in a ball left of a. Formally:
$\ell < a$ and $\exists \eta > 0, \forall x \in [a - \eta, a), f(x) > 0$

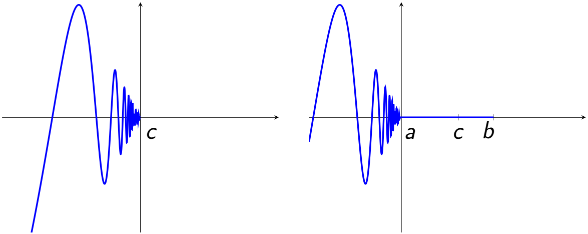
- examples: $(x-a)^{2k}$, $|x-a|^k$, $\exp\left(-\frac{1}{(x-a)^2}\right)$
- essentially $\lim_{x \to a^-} f(x) = 0^+$



Case distinction: 7 possibilities for $x < a$

Case 3: f is strictly positive and strictly negative in any ball left of a. Formally:
$\ell < a$ and $\forall \eta > 0, \exists x, y \in [a - \eta, a)$ such that $f(x) > 0$ and $f(y) < 0$

- examples: $(x-a)^k \sin\left(\frac{1}{x-a}\right)$, $\exp\left(\frac{1}{(x-a)^2}\right) \sin\left(\frac{1}{x-a}\right)$
- we will write this $\lim_{x \to a^-} f(x) = 0_\sim$



Case distinction: 7 possibilities for $x < a$

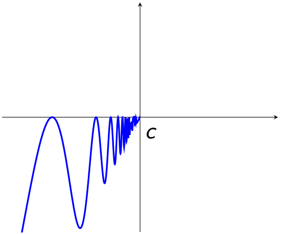
Case 4: f is negative **or** zero in a ball left of a, and is both negative **and** zero in any ball left of a. Formally:

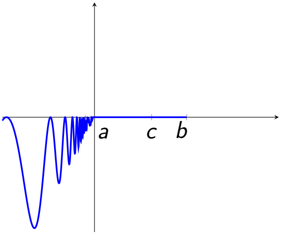
$\ell < a$ and $\exists \eta > 0, \forall x \in [a-\eta, a), f(x) \leq 0$ and

$\forall \eta > 0, \exists x, y \in [a-\eta, a)$ such that $f(x) < 0$ and $f(y) = 0$.

- examples: $-(x-a)^{2k} \sin^2\left(\frac{1}{x-a}\right)$,

  $-\exp\left(-\frac{1}{(x-a)^2}\right) \sin^2\left(\frac{1}{x-a}\right)$
- we will write this $\lim_{x \to a^-} f(x) = 0^-_\sim$

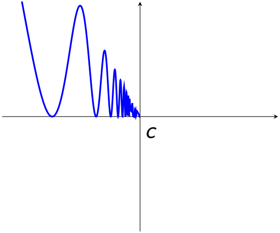


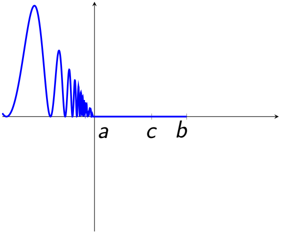


Case distinction: 7 possibilities for $x < a$

Case 5: f is positive **or** zero in a ball left of a, and is both positive **and** zero in any ball left of a. Formally:
$\ell < a$ and $\exists \eta > 0, \forall x \in [a-\eta, a), f(x) \geq 0$ and
$\forall \eta > 0, \exists x, y \in [a-\eta, a)$ such that $f(x) > 0$ and $f(y) = 0$.

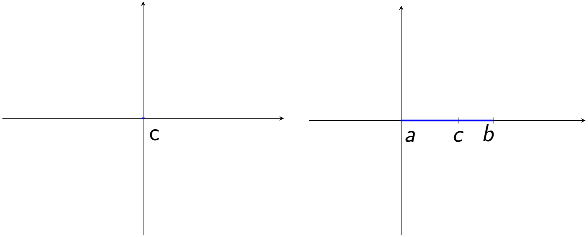
- examples: $(x-a)^{2k} \sin^2\left(\frac{1}{x-a}\right)$, $\exp\left(-\frac{1}{(x-a)^2}\right) \sin^2\left(\frac{1}{x-a}\right)$
- we will write this $\lim_{x \to a^-} f(x) = 0^+_\sim$



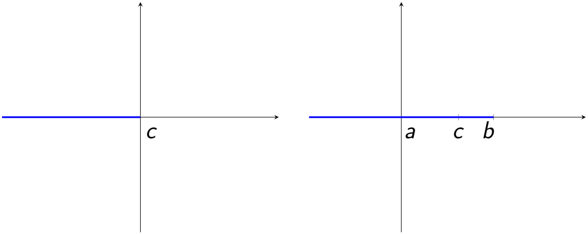


Case distinction: 7 possibilities for $x < a$

Case 6: $a = \ell \in \mathbb{R}$. Same as $\forall x \in [\ell; b\rrparenthesis, x = 0$

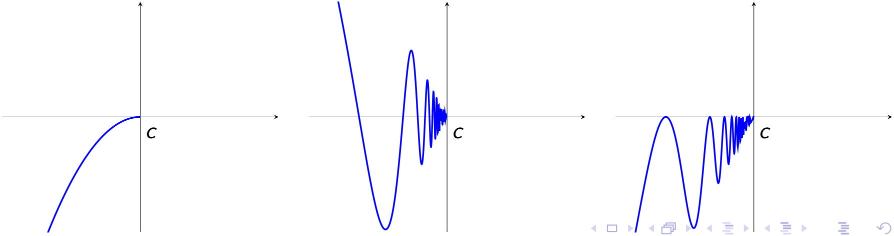


Case 7: $a = \ell = -\infty$, i.e., stays at 0 since $-\infty$



Case distinction summary: 7 possibilities for $x < a$

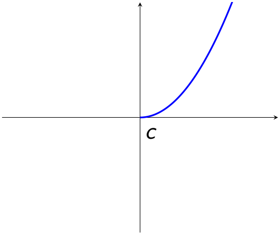
1. $\lim_{x \to a^-} f(x) = 0^-$ "strict"
2. $\lim_{x \to a^-} f(x) = 0^+$ "strict"
3. $\lim_{x \to a^-} f(x) = 0_\sim$
4. $\lim_{x \to a^-} f(x) = 0^-_\sim$ "non-strict"
5. $\lim_{x \to a^-} f(x) = 0^+_\sim$ "non-strict"
6. $a = \ell \in \mathbb{R}$ (starts from a real at zero)
7. $a = \ell = -\infty$, i.e., stays at 0 since $-\infty$

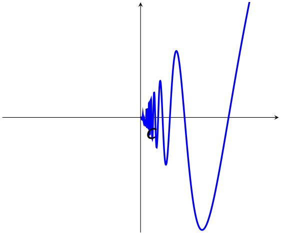


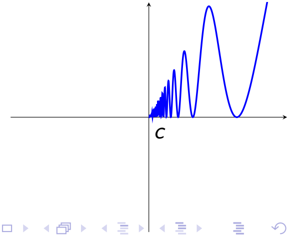
Symmetrically: 7 possibilities for $x > b$

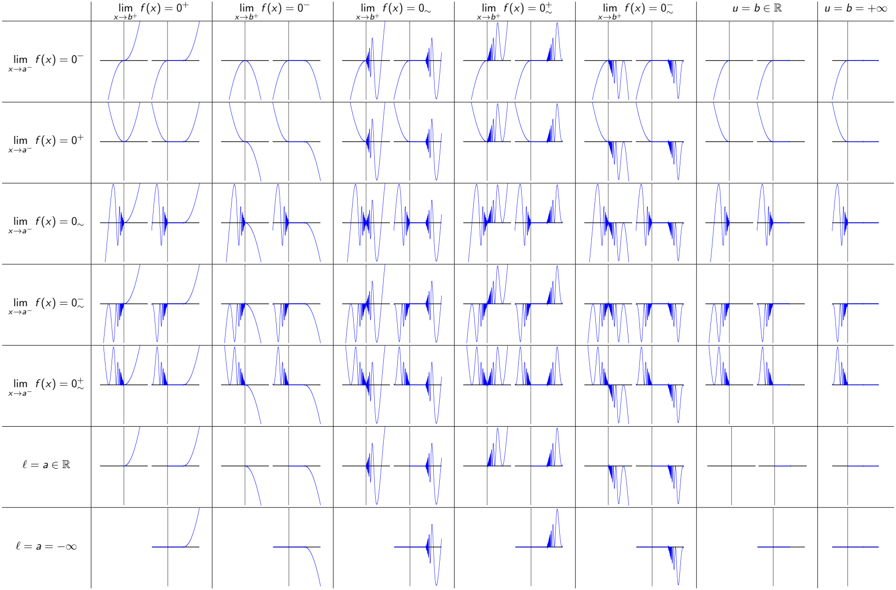
1. $\lim_{x \to b^+} f(x) = 0^+$ "strict"
2. $\lim_{x \to b^+} f(x) = 0^-$ "strict"
3. $\lim_{x \to b^+} f(x) = 0^\sim$
4. $\lim_{x \to b^+} f(x) = 0^+_\sim$ "non-strict"
5. $\lim_{x \to b^+} f(x) = 0^-_\sim$ "non-strict"
6. $b = u \in \mathbb{R}$ (ends at a real at zero)
7. $b = u = +\infty$, i.e., stays at 0 until $+\infty$

Hence 49 cases, times 2 (stays at zero or not). In reality only 85 cases.





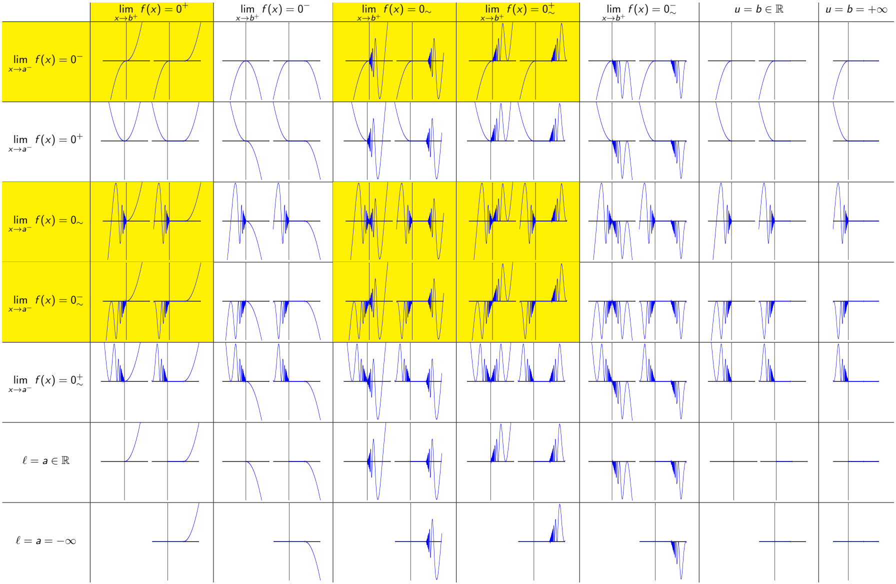




$\lim_{x\to b^+} f(x) = 0^+$
$\lim_{x\to b^+} f(x) = 0^-$
$\lim_{x\to b^+} f(x) = 0_\sim$
$\lim_{x\to b^+} f(x) = 0^+_\sim$
$\lim_{x\to b^+} f(x) = 0^-_\sim$
$u = b \in \mathbb{R}$
$u = b = +\infty$
$\lim_{x\to a^-} f(x) = 0^-$
$\lim_{x\to a^-} f(x) = 0^+$
$\lim_{x\to a^-} f(x) = 0_\sim$
$\lim_{x\to a^-} f(x) = 0^-_\sim$
$\lim_{x\to a^-} f(x) = 0^+_\sim$
$\ell = a \in \mathbb{R}$
$\ell = a = -\infty$

Desirable properties

1. If there exist $x, y \in (\!|\ell; u|\!)$ such that $x < y$, $f(x) < 0$ and $f(y) > 0$, then there exists a zero-crossing $z \in (x; y)$.
2. Its converse: The function f should be strictly negative somewhere left of, and strictly positive somewhere right of a zero-crossing z, formally:
   $\exists x, y \in (\!|\ell; u|\!), x < z < u,\ f(x) < 0$ and $f(y) > 0$.
3. A passing case (which crosses zero cleanly) should be a zero-crossing if and only if its corresponding staying case (which stays at zero for non-zero time) is one.
4. If a restriction of f to a smaller interval has a zero-crossing in z, then f should also have a zero-crossing in z.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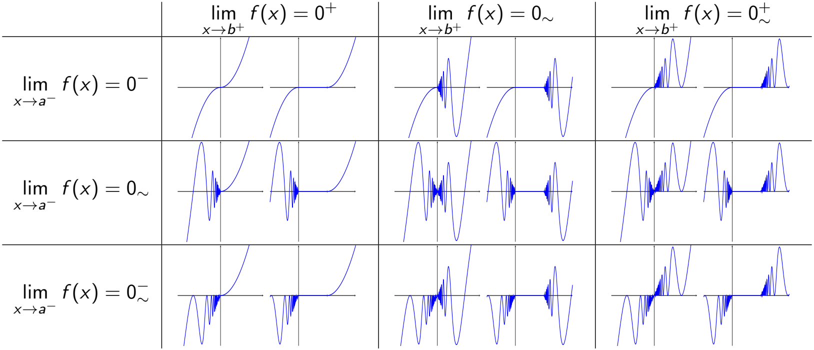
$\lim_{x\to b^+} f(x) = 0^+_\sim$
$\lim_{x\to b^+} f(x) = 0^-_\sim$
$u = b \in \mathbb{R}$
$u = b = +\infty$
$\lim_{x\to a^-} f(x) = 0^-$
$\lim_{x\to a^-} f(x) = 0^+$
$\lim_{x\to a^-} f(x) = 0_\sim$
$\lim_{x\to a^-} f(x) = 0^-_\sim$
$\lim_{x\to a^-} f(x) = 0^+_\sim$
$\ell = a \in \mathbb{R}$
$\ell = a = -\infty$

The 18 cases where $f(x) < 0$ before and $f(x) > 0$ after

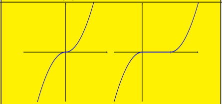
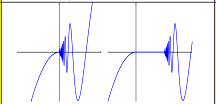
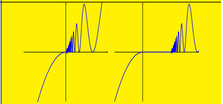
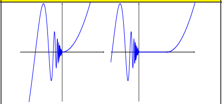
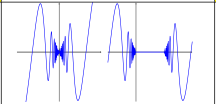
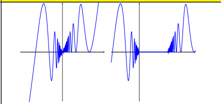
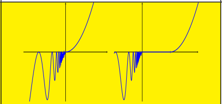
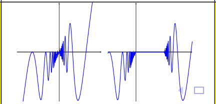
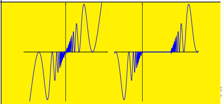


Definition attempt 1

$z \in \mathbb{R}$ is a zero-crossing for function f if and only if there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a \leq b$ and $z = b$ such that:

1. $\forall x \in [a, b], f(x) = 0;$
2. $\forall \epsilon > 0, \exists x \in [a - \epsilon; a), f(x) < 0;$
3. $\forall \epsilon > 0, \exists x \in (b; b + \epsilon], f(x) > 0;$
4. $\exists \eta > 0, \forall x \in [a - \eta; a), f(x) \leq 0;$
5. $\exists \eta > 0, \forall x \in (b; b + \eta], f(x) \geq 0.$

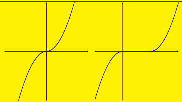
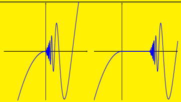
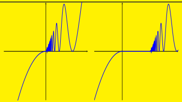
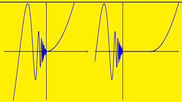
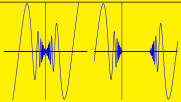
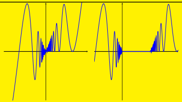
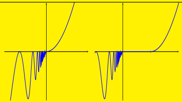
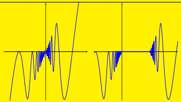
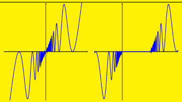
Note that in the staying case, we pick $z = b$.

	$\lim_{x \to b^+} f(x) = 0^+$	$\lim_{x \to b^+} f(x) = 0_\sim$	$\lim_{x \to b^+} f(x) = 0^+_\sim$
$\lim_{x \to a^-} f(x) = 0^-$			
$\lim_{x \to a^-} f(x) = 0_\sim$			
$\lim_{x \to a^-} f(x) = 0^-_\sim$			

Definition attempt 2

$z \in \mathbb{R}$ is a zero-crossing for function f if and only if there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a \leq b$ and $z = b$ such that:

1. $\forall x \in [a, b], f(x) = 0;$
2. $\forall \epsilon > 0, \exists x \in [a - \epsilon; a), f(x) < 0;$
3. $\forall \epsilon > 0, \exists x \in (b; b + \epsilon], f(x) > 0.$

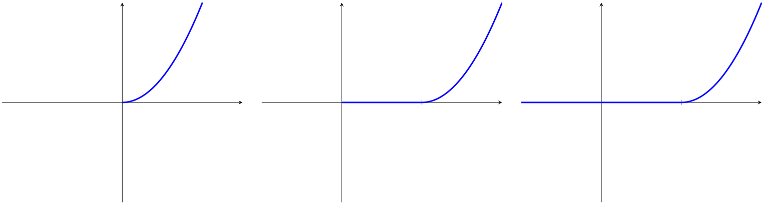
	$\lim_{x \to b^+} f(x) = 0^+$	$\lim_{x \to b^+} f(x) = 0_\sim$	$\lim_{x \to b^+} f(x) = 0^+_\sim$
$\lim_{x \to a^-} f(x) = 0^-$			
$\lim_{x \to a^-} f(x) = 0_\sim$			
$\lim_{x \to a^-} f(x) = 0^-_\sim$			

Discussion

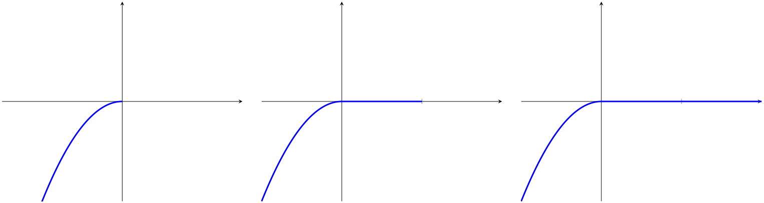
- Which definition do we like best?
- Are the desirable properties reasonable?
  1. If there exist $x, y \in (\!|\ell; u|\!)$ such that $x < y$, $f(x) < 0$ and $f(y) > 0$, then there exists a zero-crossing $z \in (x; y)$.
  2. Its converse: The function f should be strictly negative somewhere left of, and strictly positive somewhere right of a zero-crossing z, formally: $\exists x, y \in (\!|\ell; u|\!), x < z < u$, $f(x) < 0$ and $f(y) > 0$.
  3. A passing case (which crosses zero cleanly) should be a zero-crossing if and only if its corresponding staying case (which stays at zero for non-zero time) is one.
  4. If a restriction of f to a smaller interval has a zero-crossing in z, then f should also have a zero-crossing in z.

Discussion

- Would we want to include cases such as the "take-off" cases? (Note that they don't satisfy desirable properties 2 and, for some cases, 4):



- What about the symmetric to the "take-off" cases?



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